Why Volume of a Pyramid is ⅓Bh

A Visual, Calculus-Free Intuitive Proof for Secondary 2 Mathematics

No Calculus Required | Cavalieri's Slice Principle | S2 Mensuration Guide

THE MASTER FORMULA

Volume of Pyramid = ⅓ × Base Area (B) × Perpendicular Height (h)

Where B is the area of the base and h is the perpendicular distance from the top point (apex) down to the base plane.

1 The Only Tool We Need: The Slice Principle (Cavalieri's Principle)

Imagine slicing a loaf of bread or stacking coins. If you slide the stack sideways, its shape changes, but **the total amount of metal remains exactly the same!**

> **The Slice Rule:** If two 3D shapes have the *same height*, and at every horizontal slice level their cross-sectional areas match, then **their total volumes are identical.**

- If every slice of Solid A equals the slice of Solid B → **Vol A = Vol B**
- If every slice of Solid A is **k times** the slice of Solid B → **Vol A = k × Vol B**

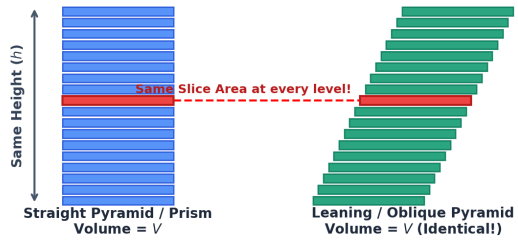
Same Height (h)
Same Slice Area at every level!
Straight Pyramid / Prism
Volume = V
Leaning / Oblique Pyramid
Volume = V (Identical!)

Figure 1: Straight vs Leaning Stack. Matching slice areas at every level guarantee identical total volume!

2 What Happens When We Slice a Pyramid? (t to t^2 Scaling)

Let's slice a pyramid parallel to its base at a fraction t of the distance down from the apex:

- At the very top (apex): $t = 0$ (Area = 0)
- At the very bottom (base): $t = 1$ (Area = B)

Because pyramid sides are straight lines, if you go depth fraction t, every length scales by t. Since **Area = Length × Width**, area scales by $t \times t = t^2$!

> Slice Area at depth $t = t^2 \times B$

Depth (t)	Length Scale	Slice Area Formula	Slice Area Fraction
$t = 0.0$ (Apex)	0×	$0^2 \times B$	**0** (Point)
$t = 0.5$ (Halfway)	½×	$(½)^2 \times B$	**¼ B**
$t = 0.75$ (¾ Down)	¾×	$(¾)^2 \times B$	**$^9/_{16}$ B**
$t = 1.0$ (Base)	1×	$1^2 \times B$	**B** (Full Base)

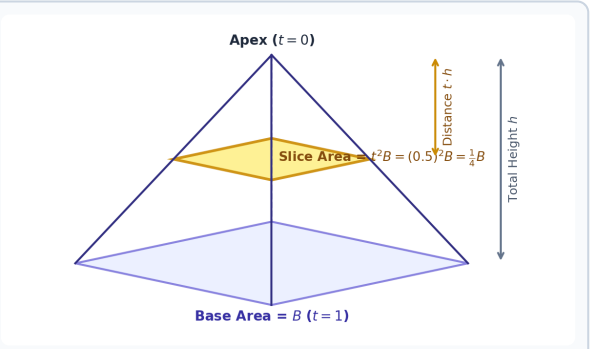

Figure 2: Slice area at depth fraction t is always t²B. At t=0.5, slice area is ¼ B.

3 Finding the Magic Factor (⅓) Using a Reference Cube

To find where the $\frac{1}{3}$ comes from, let's build one special reference pyramid inside a **cube of side length $2h$**.

Connect the center point M of the cube to all 8 outer corners. This splits the entire cube into **6 identical square pyramids**!

Reference Cube Pyramid Dimensions:

- **Height (h):** Center to face = h
- **Base Area (B_{cube}):** Face area = $(2h) \times (2h) = 4h^2$
- **Cube Volume:** $(2h)^3 = 8h^3$

Since 6 identical pyramids make up the full cube volume of $8h^3$:

$$\text{Volume of Cube Pyramid} = 8h^3 / 6 = (4/3)h^3$$

Let's check our master formula for this cube pyramid:

$$\frac{1}{3} \times \text{Base Area} \times \text{Height} = \frac{1}{3} \times (4h^2) \times h = (4/3)h^3$$

✓ **It works perfectly for the cube pyramid!**

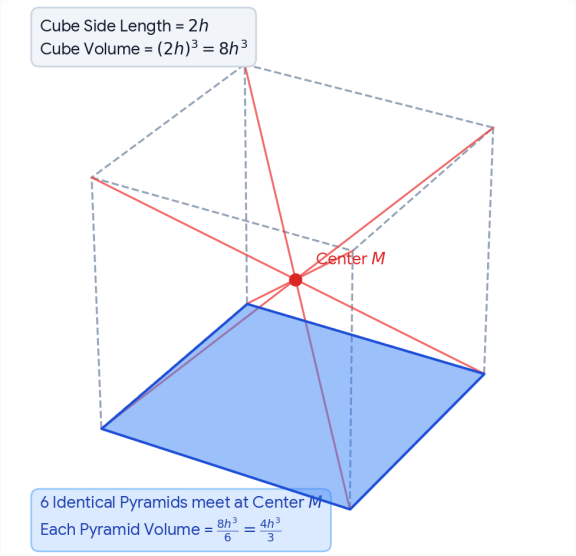

Figure 3: A cube of side 2h splits into 6 identical pyramids meeting at center M.

4 Compare ANY Pyramid with the Cube Pyramid

Now take **ANY pyramid** with height h and base area B, and compare its slices with our **Reference Cube Pyramid** (Base = $4h^2$, Height = h):

Feature	Any Arbitrary Pyramid	Reference Cube Pyramid	Ratio (Arbitrary ÷ Reference)
Height	h	h	$1 : 1$
Base Area	B	$4h^2$	$B / 4h^2$
Slice Area at depth t	$t^2 B$	$t^2 (4h^2)$	$(t^2 B) / (t^2 \cdot 4h^2) = B / 4h^2$ (t^2 cancels out!)

The Final Step: Because t^2 cancels out, at *every single height slice level*, the slice area of Any Pyramid is always exactly $B / 4h^2$ times the slice area of the Cube Pyramid!

By Cavalieri's Slice Rule, their total volumes must also be in this exact same ratio:

$$V_{any\ pyramid} = (B / 4h^2) \times V_{cube\ pyramid} = (B / 4h^2) \times ((4/3)h^3) = \frac{1}{3} Bh$$

🎓 Classroom Takeaway Summary

1. **Slice Scaling:** Slicing down fraction t scales slice area by t^2.
2. **The Cube Secret:** A cube splits into 6 identical pyramids, proving a pyramid's volume is $(4/3)h^3$.
3. **Universal Proof:** Slicing ratios prove that *every* pyramid with base B and height h has volume $V = \frac{1}{3}Bh$, whether square, triangular, or leaning!